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Multipliers of $A_p((0,\infty))$ with order convolution

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Abstract. The aim of this paper is to study the multipliers from $A_r(I)$ to $A_p(I), r \neq p$, where $I = (0,\infty)$ is the locally compact topological semigroup with multiplication max and usual topology and $A_r(I) = \{f \in L_1(I)\colon \hat{f} \in L_r(\hat{I})\}$ with norm $|||f|||_r = \|f\|_1 + \|\hat{f}\|_r$.

Keywords. Multiplier; Banach algebra; Gelfand transform.

1. Introduction

The algebra $A_p(G)$ of elements in $L_1(G)$ whose Fourier transforms belong to $L_p(\hat{G})$ and the multipliers for these algebras have been studied by various authors [1,7,8,9]. Let $I = (0,\infty)$ be the locally compact idempotent commutative topological semigroup with the usual topology and max multiplication and $\hat{I}$ be the maximal ideal space of $L_1(I)$. The algebras $A_p(I)$ of elements in $L_1(I)$ whose Gelfand transforms belong to $L_p(\hat{I})$ and the $(A_p(I), A_p(I))$ multipliers for these algebras have been studied by Kalra, Singh and Vasudeva [4]. The purpose of this note is to investigate the $(A_r(I), A_p(I))$ multipliers for $1 \leq r, p \leq \infty$. Even in the group case, the study of $(A_r(G), A_p(G))$ multipliers is not complete. Only some partial results have been listed in [2]. We have been able to obtain a set of necessary conditions and another set of sufficient conditions on a function φ so that it defines a multiplier. Some instructive examples which have bearing on the above-said necessary and sufficient conditions have been provided. The next section contains all the preliminary results which we shall use throughout the paper. The last section contains the results on the multipliers.

2. Preliminaries

Let $I = (0,\infty)$ be the locally compact semigroup with multiplication max and usual topology. Let $M(I)$ denote the Banach algebra of all finite regular Borel measures on I under the order convolution product denoted by $*$ and total variation norm. Then the Banach space $L_1(I)$ of all measures in $M(I)$ which are absolutely continuous with respect to the Lebesgue measure on I becomes a commutative semisimple Banach algebra in the inherited product $*$. More specifically, for $f, g \in L_1(I)$,

$$f * g(x) = f(x) \int_0^x g(y)\mathrm{d}y + g(x) \int_0^x f(y)\mathrm{d}y \ \text{ a.e.}$$

The maximal ideal space $\hat{I}$ of $L_1(I)$ can be identified with the interval $(0,\infty]$ and the Gelfand transform $\hat{f}$ of an $f \in L_1(I)$ is the indefinite integral, i.e.,

$$\hat{f}(x) = \int_0^x f(t)\mathrm{d}t.$$

For these and other results that may be used in the sequel, the reader is referred to [3,5].

The algebras $A_p(I), 1 \le p \le \infty$, consist of $f \in L_1(I)$ such that $\hat{f} \in L_p(\hat{I})$. Clearly $A_\infty(I) = L_1(I)$ and each $A_p(I)$ is an ideal in $L_1(I)$. Define

$$|||f|||_p = \|f\|_1 + \|\hat{f}\|_p, \quad f \in A_p(I).$$

Then $|||\cdot|||_p$ is a norm on $A_p(I)$ and $A_p(I)$ is a commutative Banach algebra with order convolution. Moreover, $A_p(I)$ is a proper subset of $A_r(I), 1 \le p < r \le \infty$. The maximal ideal space $\Delta(A_p(I))$ of $A_p(I)$ is homeomorphic to I, (note the contrast with the group case $\Delta(A_p(G)) = \Delta(L_1(G)) \cong \hat{G}$, whereas in the case of semigroup I under consideration $\Delta(L_1(I)) = \hat{I} \cong (0,\infty] \ne \Delta(A_p(I))$) and $A_p(I)$ is a semisimple commutative Banach algebra which does not contain a bounded approximate identity for $1 \le p < \infty$ but contains an approximate identity. For the above results on $A_p(I)$, the reader is referred to [4].

A mapping T on a commutative Banach algebra X to itself is called a multiplier if $T(xy) = T(x)y$ for $x, y \in X$. If X is semisimple and $T\colon X \to X$ is a multiplier then there exists a unique continuous and bounded function φ on $\Delta(X)$ such that $(\hat{T}x) = \varphi\hat{x}, x \in X$ and $\|\varphi\|_\infty \le \|T\|$ (p. 19 of [6]). Due to semisimplicity of $X, \hat{X}$ is also a commutative semisimple Banach algebra under pointwise operations with norm $\|\hat{x}\| = \|x\|$ and $\Delta(\hat{X})$ is homeomorphic to $\Delta(X)$. Therefore, for semisimple Banach algebras we may consider the multiplier T to be an operator $M_\varphi\colon \hat{X} \to \hat{X}, \hat{x} \in \hat{X}$ defined by $M_\varphi(\hat{x}) = \varphi\hat{x}, \hat{x} \in \hat{X}$. Now $A_p(I)$ is semisimple, so $\hat{A}_p(I) = \{\hat{f}\colon f \in A_p(I)\}$ is the Banach algebra under pointwise operations and $|||\hat{f}||| = |||f|||_p = \|f\|_1 + \|\hat{f}\|_p$. A multiplier from $A_r(I)$ to $A_p(I)$ may be defined (p. 67 of [6]) as a continuous function φ on $\Delta(A_r(I)) = \Delta(A_p(I)) \cong I$ such that $\varphi\hat{f} \in \hat{A}_p(I)$ whenever $\hat{f} \in \hat{A}_r(I)$.

Let A denote the algebra of all complex-valued measurable functions on I under pointwise operations. For $\varphi \in A$, let M_φ denote the operator defined by $M_\varphi(\hat{f}) = \varphi\hat{f}, \hat{f} \in \hat{A}_p(I)$. The following theorem has been proved in [4].

Theorem 1. *M_φ is a bounded multiplier on $\hat{A}_p(I)$ to itself iff*

(i) *φ is bounded*,
(ii) *φ is absolutely continuous on $[0,K]$ for each $K > 0$*,
(iii) *$M_{\varphi'}$ is a bounded linear operator from $\hat{A}_p(I)$ to $L_1(I)$.*

In the next section, we provide a characterization of multipliers from $A_r(I)$ to $A_p(I), r \ne p$. We give some intrinsic conditions on φ so that φ defines a multiplier from $A_r(I)$ to $A_p(I)$. Finally, we show that if φ is a multiplier then φ satisfies some growth conditions.

3. The $(A_r(I), A_p(I))$ multipliers

PROPOSITION 2.

If $f \in A_p(I)$ and $|||f|||_p \le 1$, then $f \in A_r(I), r > p$ and $|||f|||_r < 2$.

Proof. If $f \in A_p(I)$ and $|||f|||_p = \|f\|_1 + \|\hat{f}\|_p \le 1$, then $\|f\|_1 < 1$ and $\|\hat{f}\|_p < 1$ and

$$\begin{aligned}\|\hat{f}\|_r^r = \int_0^\infty |\hat{f}|^r &= \int_0^\infty |\hat{f}|^{r-p} \cdot |\hat{f}|^p \\ &\le \int_0^\infty |\hat{f}|^p \ \text{ as } \ |\hat{f}(x)| \le \|f\|_1 < 1 \ \text{ and } \ r-p>0 \\ &= \|\hat{f}\|_p^p < 1.\end{aligned}$$

Thus $\|\hat{f}\|_r < 1$ and $|||f|||_r = \|f\|_1 + \|\hat{f}\|_r < 2$. □

Theorem 3. *Let $M_\varphi\colon \hat{A}_r(I) \to \hat{A}_p(I), r > p$ be a bounded multiplier, then*

(i) *φ is bounded,*
(ii) *φ is absolutely continuous on $[0,K]$ for each $K > 0$ and*

$$\int_0^x |\varphi'(t)|\mathrm{d}t = O(x^{1/r}) \quad \text{as } x \to \infty,$$

(iii) *$M_{\varphi'}$ is a bounded linear operator from $\hat{A}_r(I)$ to $L_1(I)$.*

Proof. Since $r > p, A_p(I) \subseteq A_r(I)$. Therefore if $M_\varphi\colon \hat{A}_r(I) \to \hat{A}_p(I)$ is a bounded multiplier then M_φ defines a bounded multiplier from $\hat{A}_r(I)$ to itself. The theorem now follows from Theorems 15 and 18 of [4]. It should be noted that $\|\varphi\|_\infty \le 2\|M_\varphi\|$. To see this, let

$$K_x = \sup_{|||\hat{f}|||_p \le 1} |\hat{f}(x)|, \quad x \in (0,\infty).$$

Then

$$K_x \le \sup_{|||\hat{f}|||_r \le 2} |\hat{f}(x)| = 2 \sup_{|||\hat{f}/2|||_r \le 1} |(\hat{f}/2)(x)| = 2 \sup_{|||\hat{f}|||_r \le 1} |\hat{f}(x)|,$$

using Proposition 2.

Now,

$$|\varphi(x)\hat{f}(x)| = |M_\varphi(\hat{f})| \le K_x \|M_\varphi\| \ |||\hat{f}|||_r.$$

Therefore, in particular,

$$|\varphi(x)| \le \inf_{|||\hat{f}|||_r \le 1} \frac{K_x\|M_\varphi\|}{|\hat{f}(x)|} = \frac{K_x\|M_\varphi\|}{\sup_{|||\hat{f}|||_r \le 1} |\hat{f}(x)|} \le \frac{2K_x\|M_\varphi\|}{K_x} = 2\|M_\varphi\|.$$

□

The next theorem gives sufficient conditions on φ so that φ defines a multiplier from $A_r(I)$ to $A_p(I), r > p$.

Theorem 4. *Let $r > p$ and $1/v = 1/p - 1/r > 0$. If*

(i) $\varphi \in L_v(I)$,
(ii) φ *is absolutely continuous on every interval* $[0,K], K > 0$,
(iii) $M_{\varphi'}$ *is a bounded multiplier on* $\hat{A}_r(I)$ *to* $L_1(I)$,

then $M_\varphi\colon \hat{A}_r(I) \to \hat{A}_p(I)$ *is a bounded multiplier.*

Proof. Suppose φ satisfies (i), (ii) and (iii). By (i), $\lim_{x\to\infty} \varphi(x) = 0$ so there exists K such that for all $x > K, |\varphi(x)| < 1$. Using (ii) we get φ to be bounded on $[0,K]$. Thus φ is bounded. Since φ and $\hat{f}$ are absolutely continuous on $[0,K]$ for each $K > 0$ so is $\varphi\hat{f}$. Thus the derivative of $\varphi\hat{f}, (\varphi\hat{f})'$ exists a.e. on I and $(\varphi\hat{f})' = \varphi'\hat{f} + \varphi f \in L_1(I)$ in view of (iii) and boundedness of φ. Moreover,

$$\begin{aligned}\|(\varphi\hat{f})'\|_1 &= \|\varphi'\hat{f} + \varphi f\|_1 \\ &\le \|M_{\varphi'}\|\,|||\hat{f}|||_r + \|\varphi\|_\infty \|f\|_1.\end{aligned}$$

Since $\varphi \in L_v(I)$ and $\hat{f} \in L_r(I)$, it follows that $\varphi\hat{f} \in L_p(I)$ and

$$\|\varphi\hat{f}\|_p \le \|\varphi\|_v \|\hat{f}\|_r.$$

Thus $\varphi\hat{f} \in \hat{A}_p(I)$ and

$$\begin{aligned}|||\varphi\hat{f}|||_p &= \|(\varphi\hat{f})'\|_1 + \|\varphi\hat{f}\|_p \\ &\le (\|M_{\varphi'}\| + \|\varphi\|_\infty + \|\varphi\|_v)|||f|||_r.\end{aligned}$$

M_φ is a bounded multiplier on $\hat{A}_r(I)$ to $\hat{A}_p(I)$ and

$$\|M_\varphi\| \le \|M_{\varphi'}\| + \|\varphi\|_\infty + \|\varphi\|_v.$$

This completes the proof. □

Note that (iii) may be replaced by $\varphi'|_{(a,\infty)} \in L_q(a,\infty)$ for some $a > 0$ and some $q, 1 \le q \le r'$. In that case

$$\hat{f} \in L_r(a,\infty) \subseteq L_{q'}(a,\infty)$$

and

$$\|\hat{f}|_{(a,\infty)}\|_{q'}^{q'} \le \|\hat{f}|_{(a,\infty)}\|_r^r \cdot \|\hat{f}\|_\infty^{q'-r} \le \|\hat{f}|_{(a,\infty)}\|_r^r \cdot \|f\|_1^{q'-r}$$

so that $\|M_\varphi\| \le \|\varphi'|_{[0,a]}\|_1 + \|\varphi'|_{(a,\infty)}\|_q + \|\varphi\|_\infty + \|\varphi\|_v$.

Example 5.

(i) For $r > p \ge 1, 1/v = 1/p - 1/r$. Take $\varphi(x) = 1, x \in I$. Then φ is bounded and continuous on I but $\varphi \notin L_v(I), \varphi' = 0 \in L_1(I)$. So (ii) and (iii) hold but (i) does not hold in Theorem 4. For

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ -\alpha x^{-\alpha-1}, & x \ge 1, \end{cases}$$

$$\hat{f}(x) = \begin{cases} x, & 0 < x < 1, \\ x^{-\alpha}, & x \geq 1. \end{cases}$$

For $0 < 1/r < \alpha < 1/p$ (α exists as $r > p$), $\hat{f} \in L_r(I) \backslash L_p(I)$, so $\varphi \hat{f} = \hat{f} \notin L_p(I)$. Thus φ is not a multiplier on $\hat{A}_r(I)$ to $\hat{A}_p(I)$.

(ii) If $r > p \geq 1, 1/v = 1/p - 1/r > 0$ and $\varepsilon > 0$, take

$$\varphi(x) = \begin{cases} 1, & 0 < x \leq 1, \\ x^{-\frac{1}{v}-\varepsilon}, & x > 1. \end{cases}$$

Then φ is continuous on $I, \varphi \in L_v(I)$, φ is absolutely continuous on $[0,K]$ for all $K > 0$.

$$\varphi'(x) = \begin{cases} 0, & 0 < x < 1, \\ \left(-\frac{1}{v} - \varepsilon\right) x^{-\frac{1}{v}-\varepsilon-1}, & x > 1, \end{cases}$$

$$= \begin{cases} 0, & 0 < x < 1, \\ \left(-\frac{1}{v} - \varepsilon\right) x^{-\frac{1}{r'}-\frac{1}{p}-\varepsilon}, & x > 1, \end{cases}$$

is in $L_{r'}(1,\infty)$ so that conditions of Theorem 4 are satisfied. Hence M_φ is a multiplier from $\hat{A}_r(I)$ to $\hat{A}_p(I)$.

In the next two theorems, we discuss the case $r < p$.

Theorem 6. *If*

(i) *φ is bounded on $(0,\infty)$ or $\varphi \in L_p(I)$,*
(ii) *φ is absolutely continuous on $[0,K]$ for each $K > 0$,*
(iii) *$M_{\varphi'}$ is a bounded multiplier on $\hat{A}_r(I)$ to $L_1(I)$, then $M_\varphi \colon \hat{A}_r(I) \to \hat{A}_p(I), r < p$ is a bounded multiplier.*

Proof. If φ is bounded on I and (ii) and (iii) hold, then $M_\varphi \colon \hat{A}_r(I) \to \hat{A}_r(I)$ is a bounded multiplier by Theorem 1. So $M_\varphi \colon \hat{A}_r(I) \to \hat{A}_p(I)$ is a bounded multiplier as $\hat{A}_r(I) \subseteq \hat{A}_p(I)$ for $r < p$. Moreover,

$$\begin{aligned} \|M_\varphi\| &= \sup_{|||\hat{f}|||_r = 1} (|||\varphi \hat{f}|||_p) \\ &= \sup_{|||\hat{f}|||_r = 1} (\|\varphi \hat{f}\|_p + \|\varphi' \hat{f} + \varphi f\|_1) \\ &\leq \sup_{|||\hat{f}|||_r = 1} (\|\varphi\|_\infty \|\hat{f}\|_p + \|M_{\varphi'}\| \, |||\hat{f}|||_r + \|\varphi\|_\infty \|f\|_1) \\ &\leq 2\|\varphi\|_\infty + \|M_{\varphi'}\| \text{ as } \|f\|_1 < 1 \text{ and } \|\hat{f}\|_p < 1 \text{ using Proposition 2.} \end{aligned}$$

Next, if $\varphi \in L_p(I)$, then as in the proof of Theorem 4, φ is bounded on I and hence $M_\varphi \colon \hat{A}_r(I) \to \hat{A}_r(I)$ is a bounded multiplier (using Theorem 1). □

Theorem 7. *If M_φ is a bounded multiplier on $\hat{A}_r(I)$ to $\hat{A}_p(I), r < p$ then φ is continuous on I, φ is absolutely continuous on $[0,K]$ for $K > 0, \varphi$ is locally in $L_p(I)$ and*

$$\|\varphi|_{[0,x]}\|_p = o\left(x^{\frac{1}{r}+\varepsilon}\right),$$

$$\|\varphi'|_{[0,x]}\|_1 = o\left(x^{\frac{1}{r}+\varepsilon}\right) \quad \text{for all } \varepsilon > 0.$$

Proof. The proof is exactly similar to that of the proof of theorem 15 of [4]. Let $M_\varphi : \hat{A}_r(I) \to \hat{A}_p(I), r < p$ be a bounded multiplier, then φ is continuous and

$$|||\varphi \hat{f}|||_p = |||M_\varphi(\hat{f})|||_p \le \|M_\varphi\| \; |||\hat{f}|||_r, \quad f \in A_r(I).$$

Take the function

$$f(x) = \begin{cases} 1/\alpha, & 0 < x < \alpha, \\ 0, & \alpha \le x < \beta, \\ 1/(\beta - \gamma), & \beta \le x < \gamma, \\ 0, & x \ge \gamma. \end{cases}$$

As in the proof of theorem 15 of [4], $\varphi \hat{f} \in \hat{A}_p(I)$ implies φ is continuous on $(0,\infty), \varphi'$ exists a.e. on $(0,\infty)$ and

$$\|\varphi \hat{f}\|_p + \|\varphi' \hat{f} + \varphi f\|_1 = |||\varphi \hat{f}|||_p \le \|M_\varphi\| \; |||\hat{f}|||_r.$$

This implies that

$$\|\varphi|_{[\alpha,\beta]}\|_p + \|\varphi'|_{[\alpha,\beta]}\|_1 \le \|M_\varphi\| \left[2 + \left(\frac{\gamma + r(\beta - \alpha)}{r+1}\right)^{1/r}\right].$$

Fix β and vary $\gamma > \beta$. Then

$$\sup_{0<\alpha<\beta} (\|\varphi|_{[\alpha,\beta]}\|_p + \|\varphi'|_{[\alpha,\beta]}\|_1) \le \|M_\varphi\|(2 + \beta^{1/r}).$$

So $\|\varphi|_{[0,\beta]}\|_p = o(\beta^{\frac{1}{r}+\varepsilon})$ and $\|\varphi'|_{[0,\beta]}\|_1 = o(\beta^{\frac{1}{r}+\varepsilon})$ for all $\varepsilon > 0$. Thus φ is absolutely continuous on $[0,\beta]$ for $\beta > 0$. This completes the proof. □

Example 8. For $\varepsilon > 0$ and $r < p$, consider

$$\varphi(x) = \begin{cases} 1, & 0 < x < 1, \\ x^{\frac{1}{r} - \frac{1}{p} + \varepsilon}, & x \ge 1. \end{cases}$$

Then

$$\varphi'(x) = \begin{cases} 0, & 0 < x < 1, \\ \left(\frac{1}{r} - \frac{1}{p} + \varepsilon\right) x^{\frac{1}{r} - \frac{1}{p} + \varepsilon - 1}, & x \ge 1, \end{cases}$$

φ is continuous on I, φ' exists a.e. on I and

$$\|\varphi'|_{[0,\beta]}\|_1 = \beta^{\frac{1}{r}-\frac{1}{p}+\varepsilon} - 1 = o\left(\beta^{\frac{1}{r}+2\varepsilon}\right)$$

and

$$\|\varphi|_{[0,\beta]}\|_p = \left(1+\frac{\beta^{p/r+p\varepsilon}-1}{p/r+p\varepsilon}\right)^{1/p} = o\left(\beta^{\frac{1}{r}+2\varepsilon}\right) \quad \text{for } \varepsilon > 0.$$

Thus φ satisfies the necessary conditions of Theorem 7 for being a multiplier from $\hat{A}_r(I)$ to $\hat{A}_p(I), r < p$. But φ is not bounded and $\varphi \notin L_p(I)$. Also $\varphi' \in L_{r'}(I)$ for $0 < \varepsilon < 1/p$. It follows that $M_{\varphi'}(I)$ is a bounded multiplier from $\hat{A}_r(I)$ to $L_1(I)$ and (iii) holds in Theorem 6.

For

$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ \left(-\frac{1}{r}-\varepsilon\right) x^{-\frac{1}{r}-\varepsilon-1}, & x \geq 1, \end{cases}$$

$$\hat{f}(x) = \begin{cases} x, & 0 < x < 1, \\ x^{-\frac{1}{r}-\varepsilon}, & x \geq 1, \end{cases}$$

$f \in A_r(I)$, but

$$\varphi\hat{f}(x) = \begin{cases} x, & 0 < x < 1, \\ x^{-\frac{1}{p}}, & x \geq 1, \end{cases}$$

is not in $L_p(I)$. So M_φ is not a multiplier from $\hat{A}_r(I)$ to $\hat{A}_p(I)$.

Note. All the multipliers from $\hat{A}_r(I)$ to $\hat{A}_p(I), r < p$, will not be given by continuous bounded functions φ otherwise as in the proof of theorem 1 bounded multipliers from $\hat{A}_r(I)$ to $\hat{A}_p(I), r < p$ will only be the multipliers from $\hat{A}_r(I)$ to $\hat{A}_r(I) \subset \hat{A}_p(I)$.

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